Supporting Information

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SI Appendix

1. Expression of the Growth Rate Λ for the General Model

The general model is defined by the following:

$$\begin{aligned} \gamma' &= \lambda\gamma + \kappa z_t + \omega\phi + \nu_H, \quad \nu_H \sim \mathcal{N}\left(0, \sigma_H^2\right), \\ \phi &= \theta\gamma + \rho y_t + \nu_D, \quad \nu_D \sim \mathcal{N}\left(0, \sigma_D^2\right), \\ \langle\xi\rangle_{\gamma,x_t} &= \exp\left[r_{\max} - (\phi - x_t)^2 / \left(2\sigma_S^2\right)\right], \end{aligned} \qquad \textbf{[S1]}$$

where γ' represents the transmitted genotype, γ the inherited genotype, ϕ the current phenotype, x_t the environment, and $\langle\xi\rangle_{\gamma,x_t}$ the expected number of offsprings for individuals with genotype γ in environment x_t. The variables y_t and z_t represent possible cues coming from the environment x_t and are given by the following:

$$\begin{aligned} x_t &= a x_{t-1} + b_t, \quad b_t \sim \mathcal{N}\left(0, \sigma_X^2\right), \\ y_t &= x_t + b'_t, \quad b'_t \sim \mathcal{N}\left(0, \sigma_{Ip}^2\right), \\ z_t &= x_t + b''_t, \quad b''_t \sim \mathcal{N}\left(0, \sigma_{I\ell}^2\right). \end{aligned} \qquad \textbf{[S2]}$$

The simple model presented in the first part of the main text corresponds to $\lambda = 1$, $\theta = 1$, $\sigma_D^2 = 0$, $\kappa = 0$, $\omega = 0$, $\rho = 0$. Assuming that all variances are expressed in units of σ_S^2, or, equivalently, assuming that $\sigma_S^2 = 1$, an analytical expression for the long-term growth rate Λ of the general model is the following:

$$\begin{aligned} \Lambda = r_{\max} &+ \frac{1}{2}\ln\left(\frac{\alpha}{\tilde\lambda\left(\sigma_D^2+1\right)}\right) - \frac{1}{2}\frac{\alpha}{\tilde\lambda\left(\sigma_D^2+1\right)} \\ &\times\left[\left(1-\rho^2\right)\frac{(1+a\alpha)\left(1+\left(\tilde\lambda+\tilde\kappa\right)^2\right)-2(a+\alpha)\left(\tilde\lambda+\tilde\kappa\right)}{(1-a\alpha)\left(1-\alpha^2\right)}\sigma_E^2\right. \\ &\left.+\rho^2\frac{1+\left(\tilde\lambda+\tilde\omega\right)^2-2\alpha\left(\tilde\lambda+\tilde\omega\right)}{1-\alpha^2}\sigma_{Ip}^2+\kappa^2\theta^2\sigma_{I\ell}^2\right], \end{aligned} \qquad \textbf{[S3]}$$

where

$$\begin{aligned} \alpha &= \frac{2\tilde\lambda}{1+\tilde\lambda^2+\tilde\sigma_H^2+\left(\left(1-\tilde\lambda^2-\tilde\sigma_H^2\right)^2+4\tilde\sigma_H^2\right)^{1/2}}, \\ \tilde\sigma_H^2 &= \left(\sigma_H^2+\frac{\omega^2\sigma_D^2}{\sigma_D^2+1}\right)\frac{\theta^2}{\sigma_D^2+1}, \\ \tilde\lambda &= \lambda+\frac{\theta\omega}{\sigma_D^2+1}, \qquad \tilde\kappa = \left(\frac{\sigma_D^2+\rho}{\sigma_D^2+1}\omega+\kappa\right)\frac{\theta}{1-\rho}, \\ \tilde\omega &= \frac{\left(\sigma_D^2-1\right)\theta\omega}{\left(\sigma_D^2+1\right)(1-\rho)}. \end{aligned} \qquad \textbf{[S4]}$$

The derivation of this formula is presented below. We verified its validity by comparisons with numerical simulations of the population dynamics.

2. Derivations of the Formula for Λ

We detail here the derivation of the analytical formula for the growth rate Λ of the general model, Eq. **S3**. The derivation relies on the stability of Gaussian functions under multiplication and convolution. Denoting by $G_{\sigma^2}(x)$ a generic Gaussian function, i.e.,

$$G_{\sigma^2}(x) \equiv \frac{1}{\left(2\pi\sigma^2\right)^{1/2}}\exp\left(-\frac{x^2}{2\sigma^2}\right), \qquad \textbf{[S5]}$$

the property of stability under multiplication corresponds to the following identity:

$$\begin{aligned} &G_{\sigma_1^2}(\lambda_1 x - m_1)G_{\sigma_2^2}(\lambda_2 x - m_2) \\ &= G_{\left(\lambda_1^2\sigma_1^{-2}+\lambda_2^2\sigma_2^{-2}\right)^{-1}}\left(x - \frac{\lambda_1\sigma_2^2}{\lambda_2^2\sigma_1^2+\lambda_1^2\sigma_2^2}m_1 - \frac{\lambda_2\sigma_1^2}{\lambda_2^2\sigma_1^2+\lambda_1^2\sigma_2^2}m_2\right) \\ &\quad\times G_{\lambda_2^2\sigma_1^2+\lambda_1^2\sigma_2^2}(\lambda_2 m_1 - \lambda_1 m_2). \end{aligned} \qquad \textbf{[S6]}$$

This identity implies the stability under convolution:

$$\int \mathrm{d}x\; G_{\sigma_1^2}(\lambda_1 x - m_1)G_{\sigma_2^2}(\lambda_2 x - m_2) = G_{(\lambda_2\sigma_1)^2+(\lambda_1\sigma_2)^2}(\lambda_2 m_1 - \lambda_1 m_2). \qquad \textbf{[S7]}$$

2.1. Particular Model. We first treat a simpler model, to which the general model maps once properly parameterized. This particular model, obtained from the general model by setting $\sigma_D^2 = 0$, $\theta = 1$, $\rho = 0$, $\omega = 0$, is itself a slight generalization of the model described by Eq. **18** in the main text. It is defined by $A(\xi, \gamma'|\gamma, x_t) = R(\xi|\gamma, x_t)H(\gamma'|\gamma, x_t)$, with the following:

$$\begin{aligned} \sum_\xi \xi\, R(\xi|\gamma, x_t) &= \exp\left(r_{\max} - \frac{(\gamma - x_t)^2}{2\sigma_S^2}\right), \\ H\left(\gamma'|\gamma, x_t\right) &= G_{\sigma_H^2}\left(\gamma' - \lambda\gamma - \kappa x_t\right). \end{aligned} \qquad \textbf{[S8]}$$

Considering an environment following $x_{t+1} = a x_t + b_t$ with $b_t \sim \mathcal{N}(0, \sigma_X^2)$, and ignoring the trivial factor $r_{\max}$ [Λ with $r_{\max} > 0$ is obtained as $\Lambda = \Lambda(r_{\max} = 0) + r_{\max}$], this model has in total six parameters: a, σ_X^2, σ_S^2, σ_H^2, λ, and κ.

We start from the recursion for the composition $n_t(\gamma)$ of the population, Eq. **1**, which we write under the assumption that $n_t(\gamma)$ is Gaussian, $n_t(\gamma) = G_{\sigma_t^2}(\gamma - m_t)$,

$$\begin{aligned} W_t\, n_{t+1}\left(\gamma'\right) &= \left(2\pi\sigma_S^2\right)^{1/2} \\ &\quad\times\int \mathrm{d}\gamma\; G_{\sigma_S^2}(\gamma - x_t)G_{\sigma_H^2}\left(\gamma' - \lambda\gamma - \kappa x_t\right)G_{\sigma_t^2}(\gamma - m_t). \end{aligned} \qquad \textbf{[S9]}$$

We first use Eq. **S6** to obtain the following:

$$W_t = \left(2\pi\sigma_S^2\right)^{1/2}G_{\sigma_S^2+\sigma_t^2}(m_t - x_t), \qquad \textbf{[S10]}$$

$$\begin{aligned} n_{t+1}\left(\gamma'\right) &= \int \mathrm{d}\gamma\; G_{\sigma_H^2}\left(\gamma' - \lambda\gamma - \kappa x_t\right) \\ &\quad\times G_{\left(\sigma_R^{-2}+\sigma_t^{-2}\right)^{-1}}\left(\gamma - \frac{\sigma_t^2}{\sigma_S^2+\sigma_t^2}x_t - \frac{\sigma_S^2}{\sigma_S^2+\sigma_t^2}m_t\right), \end{aligned} \qquad \textbf{[S11]}$$

and then Eq. **S7** to identify the mean m_{t+1} and variance σ_{t+1}^2 of $n_{t+1}(\gamma')=G_{\sigma_{t+1}^2}(\gamma'-m_{t+1})$,

$$m_{t+1}=\lambda\frac{\sigma_S^2}{\sigma_S^2+\sigma_t^2}m_t+\left(\lambda\frac{\sigma_t^2}{\sigma_S^2+\sigma_t^2}+\kappa\right)x_t, \qquad \textbf{[S12]}$$

$$\sigma_{t+1}^2=\lambda^2\left(\sigma_S^{-2}+\sigma_t^{-2}\right)^{-1}+\sigma_H^2. \qquad \textbf{[S13]}$$

Eq. **S10** can also be written as follows:

$$\ln W_t=\frac{1}{2}\ln\left(\frac{\sigma_S^2}{\sigma_S^2+\sigma_t^2}\right)-\frac{1}{2}\frac{(m_t-x_t)^2}{\sigma_S^2+\sigma_t^2}, \qquad \textbf{[S14]}$$

from which the long-term growth rate Λ is derived as $\Lambda=\lim_{t\to\infty}\mathbb{E}[\ln W_t]$. Taking $\lambda=1$ and $\kappa=0$, and introducing the notation $h_t^2\equiv\sigma_t^2/(\sigma_t^2+\sigma_S^2)$, this leads to the expression given in the main text for the simple model, Eq. **12** in the main text.

In the limit $t\to\infty$, the recursion for σ_t^2, Eq. **S13**, leads a fixed point σ_∞^2 that is solution to the following:

$$\sigma_\infty^2=\lambda^2\left(\sigma_\infty^{-2}+\sigma_S^{-2}\right)^{-1}+\sigma_H^2. \qquad \textbf{[S15]}$$

For large t, we can therefore approximate the recursion for m_t, Eq. **S12**, by

$$m_{t+1}-x_{t+1}=\alpha(m_t-x_t)-(x_{t+1}-(\lambda+\kappa)x_t), \qquad \textbf{[S16]}$$

where α, defined by

$$\alpha\equiv\frac{\lambda\sigma_S^2}{\sigma_S^2+\sigma_\infty^2}, \qquad \textbf{[S17]}$$

is, after solving Eq. **S15**, found to be the following:

$$\alpha=\frac{2\lambda}{1+\lambda^2+\beta+\left(\left(1-\lambda^2-\beta\right)^2+4\beta\right)^{1/2}},\quad \text{with}\quad \beta\equiv\frac{\sigma_H^2}{\sigma_S^2}. \qquad \textbf{[S18]}$$

With these notations, the long-term growth rate $\Lambda=\lim_{t\to\infty}\mathbb{E}[\ln W_t]$ can be written as follows:

$$\Lambda=\frac{1}{2}\ln\left(\frac{\alpha}{\lambda}\right)-\frac{1}{2}\frac{\alpha}{\lambda}\lim_{t\to\infty}\frac{\mathbb{E}\left[(m_t-x_t)^2\right]}{\sigma_S^2}. \qquad \textbf{[S19]}$$

Completing the solution requires estimating $\lim_{t\to\infty}\mathbb{E}[(m_t-x_t)^2]$. To this end, we first rewrite Eq. **S16** as follows:

$$m_{t+1}-x_{t+1}=\alpha(m_t-x_t)+\epsilon\, x_t-b_t,\quad \text{with}\quad \epsilon\equiv\lambda+\kappa-a, \qquad \textbf{[S20]}$$

which implies

$$m_{t+1}-x_{t+1}=\sum_{k=0}^{t}\alpha^{t-k}(\epsilon x_k-b_k). \qquad \textbf{[S21]}$$

Using the relation $x_k=\sum_{n=0}^{k-1}a^{k-1-n}b_n$, which follows from the recursion $x_k=ax_{k-1}+b_{k-1}$, we have the following:

$$\sum_{k=0}^{t}\alpha^{t-k}x_k=\sum_{k=0}^{t-1}\sum_{n=0}^{t-k-1}\alpha^{t-n-k-1}a^nb_k=\sum_{k=0}^{t-1}\frac{1-\left(a\alpha^{-1}\right)^{t-k}}{1-a\alpha^{-1}}\alpha^{t-(k+1)}b_k$$
$$=\sum_{k=0}^{t-1}\frac{\alpha^{t-k}-a^{t-k}}{\alpha-a}b_k, \qquad \textbf{[S22]}$$

where the last sum can be extended up to $k=t$. We thus get the following:

$$m_{t+1}-x_{t+1}=\frac{1}{\alpha-a}\sum_{k=0}^{t}\left[(\epsilon-\alpha+a)\alpha^{t-k}-\epsilon a^{t-k}\right]b_k. \qquad \textbf{[S23]}$$

Because the b_k are independent and identically distributed random variables with variance σ_X^2, we have the following:

$$\mathbb{E}\left[(m_{t+1}-x_{t+1})^2\right]=\frac{1}{(\alpha-a)^2}\sum_{k=0}^{t}\left[(\epsilon-\alpha+a)\alpha^{t-k}-\epsilon a^{t-k}\right]^2\sigma_X^2. \qquad \textbf{[S24]}$$

Summing the geometric series and taking the limit $t\to\infty$ then leads to the following:

$$\lim_{t\to\infty}\mathbb{E}\left[(m_t-x_t)^2\right]=\frac{1}{(\alpha-a)^2}\times\left(\frac{(\epsilon-\alpha+a)^2}{1-\alpha^2}-\frac{2\epsilon(\epsilon-\alpha+a)}{1-\alpha a}+\frac{\epsilon^2}{1-a^2}\right)\sigma_X^2, \qquad \textbf{[S25]}$$

or, equivalently,

$$\lim_{t\to\infty}\mathbb{E}\left[(m_t-x_t)^2\right]=\frac{(1+a\alpha)\left(1+(\epsilon+a)^2\right)-2(a+\alpha)(\epsilon+a)}{(1-a^2)(1-\alpha^2)(1-a\alpha)}\sigma_X^2. \qquad \textbf{[S26]}$$

Finally, replacing $\lim_{t\to\infty}\mathbb{E}[(m_t-x_t)^2]$ with this expression and invoking the relations $1/(\sigma_\infty^2+\sigma_S^2)=\alpha/(\lambda\sigma_S^2)$ and $\epsilon=\lambda+\kappa-a$ yields the following:

$$\Lambda=\frac{1}{2}\ln\left(\frac{\alpha}{\lambda}\right)-\frac{\alpha}{2\lambda}\frac{(1+a\alpha)\left(1+(\lambda+\kappa)^2\right)-2(a+\alpha)(\lambda+\kappa)}{(1-a^2)(1-\alpha^2)(1-a\alpha)}\frac{\sigma_X^2}{\sigma_S^2}, \qquad \textbf{[S27]}$$

which, together with Eq. **S18**, provides an analytical formula for the long-term growth rate Λ of this particular model. With $\lambda=1$ and $\kappa=0$, this formula corresponds to the expression for Λ given by Eq. **18** in the main text.

2.2. General Model. The general model is defined by $A(\xi,\gamma'|\gamma,x_t)=\int d\phi\; R(\xi|\gamma,\phi,x_t)D(\phi|\gamma,y_t)H(\gamma'|\gamma,\phi,z_t)$, with

$$\sum_\xi \xi\, R(\xi|\gamma,x_t)=\left(2\pi\sigma_S^2\right)^{1/2}G_{\sigma_S^2}(\gamma-x_t),$$
$$D(\phi|\gamma,y_t)=G_{\sigma_D^2}\left(\gamma'-\theta\gamma-\rho y_t\right),$$
$$H\left(\gamma'|\gamma,\phi,z_t\right)=G_{\sigma_H^2}\left(\gamma'-\lambda\gamma-\omega\phi-\kappa z_t\right) \qquad \textbf{[S28]}$$

(we assume here $r_{\max}=0$). All of the environmental variables, represented by roman letters, are defined independently of the dynamics of the population, by $x_{t+1}=ax_t+b_t$, $y_t=x_t+b_t'$, and $z_t=x_t+b_t''$ with $b_t\sim\mathcal{N}(0,\sigma_X^2)$, $b_t'\sim\mathcal{N}(0,\sigma_{Ip}^2)$, and $b_t''\sim\mathcal{N}(0,\sigma_{I\ell}^2)$.

Under the assumption that the attribute γ is normally distributed in the population, $n_t(\gamma)=G_{\sigma_t^2}(\gamma-m_t)$, the basic recursion defining the model is the following:

$$W_t\, n_{t+1}\left(\gamma'\right)=\left(2\pi\sigma_S^2\right)^{1/2}\int d\gamma\, d\phi\; G_{\sigma_S^2}(\phi-x_t)G_{\sigma_D^2}(\phi-\theta\gamma-\rho y_t)$$
$$\times G_{\sigma_H^2}\left(\gamma'-\lambda\gamma-\omega\phi-\kappa z_t\right)G_{\sigma_t^2}(\gamma-m_t). \qquad \textbf{[S29]}$$

The first steps that we take for solving this general model mirror those taken to solve the particular model. Using Eq. **S6**, $G_{\sigma_S^2}(\phi - x_t) G_{\sigma_D^2}(\phi - \theta\gamma - \rho y_t)$ is rewritten as $P(\phi,\gamma)X(\gamma)$ with

$$P(\phi,\gamma) = G_{\left(\sigma_D^{-2}+\sigma_S^{-2}\right)^{-1}}\left(\phi - \frac{\sigma_D^2}{\sigma_D^2+\sigma_S^2}x_t - \frac{\sigma_S^2}{\sigma_D^2+\sigma_S^2}(\theta\gamma+\rho y_t)\right),$$
$$X(\gamma) = G_{\sigma_D^2+\sigma_S^2}(\theta\gamma - (x_t - \rho y_t)). \quad \textbf{[S30]}$$

The integration over ϕ in $W_t n_{t+1}(\gamma') = (2\pi\sigma_S^2)^{1/2}\int d\gamma\, d\phi\, P(\phi,\gamma) X(\gamma) H(\gamma'|\gamma,\phi,z_t) G_{\sigma_t^2}(\gamma - m_t)$ involves only $Y(\gamma,\gamma') = \int d\phi\, P(\phi,\gamma) H(\gamma'|\gamma,\phi,z_t)$ with

$$Y(\gamma,\gamma') = G_{\omega^2\left(\sigma_D^{-2}+\sigma_S^{-2}\right)^{-1}+\sigma_H^2}\left(\left(\lambda + \frac{\sigma_S^2}{\sigma_D^2+\sigma_S^2}\omega\theta\right)\gamma - \gamma' + \frac{\sigma_D^2}{\sigma_D^2+\sigma_S^2}\omega x_t + \frac{\sigma_S^2}{\sigma_D^2+\sigma_S^2}\omega\rho y_t + \kappa z_t\right), \quad \textbf{[S31]}$$

so that $W_t n_{t+1}(\gamma') = (2\pi\sigma_S^2)^{1/2}\int d\gamma\, X(\gamma)Y(\gamma,\gamma')G_{\sigma_t^2}(\gamma - m_t)$. By rewriting $(2\pi\sigma_S^2)^{1/2}X(\gamma)G_{\sigma_t^2}(\gamma - m_t)$ as we obtain

$$Z(\gamma) = G_{\left(\theta^2\left(\sigma_D^2+\sigma_S^2\right)^{-1}+\sigma_t^{-2}\right)^{-1}}\left(\gamma - \frac{\sigma_t^2}{\sigma_D^2+\sigma_S^2+\theta^2\sigma_t^2}\theta(x_t-\rho y_t) - \frac{\sigma_D^2+\sigma_S^2}{\sigma_D^2+\sigma_S^2+\theta^2\sigma_t^2}m_t\right), \quad \textbf{[S32]}$$

and

$$W_t = (2\pi\sigma_S^2)^{1/2} G_{\sigma_D^2+\sigma_S^2+\theta^2\sigma_t^2}\left(\theta m_t - (x_t - \rho y_t)\right), \quad \textbf{[S33]}$$

i.e.,

$$\ln W_t = \frac{1}{2}\ln\left(\frac{\sigma_S^2}{\sigma_D^2+\sigma_S^2+\theta^2\sigma_t^2}\right) - \frac{\left(\theta m_t - (x_t-\rho y_t)\right)^2}{2\left(\sigma_D^2+\sigma_S^2+\theta^2\sigma_t^2\right)}. \quad \textbf{[S34]}$$

Finally, $n_{t+1}(\gamma') = \int d\gamma\, Y(\gamma,\gamma')Z(\gamma) = G_{\sigma_{t+1}^2}(\gamma' - m_{t+1})$, with

$$\sigma_{t+1}^2 = \omega^2\left(\sigma_D^{-2}+\sigma_S^{-2}\right)^{-1} + \sigma_H^2 + \left(\lambda + \frac{\sigma_S^2}{\sigma_D^2+\sigma_S^2}\omega\theta\right)^2\left(\theta^2\left(\sigma_D^2+\sigma_S^2\right)^{-1}+\sigma_t^{-2}\right)^{-1}, \quad \textbf{[S35]}$$

and

$$m_{t+1} = \left(\lambda + \frac{\sigma_S^2}{\sigma_D^2+\sigma_S^2}\omega\theta\right)\frac{\sigma_D^2+\sigma_S^2}{\sigma_D^2+\sigma_S^2+\theta^2\sigma_t^2}m_t + \left(\frac{\sigma_D^2}{\sigma_D^2+\sigma_S^2}\omega + \left(\lambda + \frac{\sigma_S^2}{\sigma_D^2+\sigma_S^2}\omega\theta\right)\frac{\sigma_t^2}{\sigma_D^2+\sigma_S^2+\theta^2\sigma_t^2}\theta\right)x_t + \left(\frac{\sigma_S^2}{\sigma_D^2+\sigma_S^2}\omega - \left(\lambda + \frac{\sigma_S^2}{\sigma_D^2+\sigma_S^2}\omega\theta\right)\frac{\sigma_t^2}{\sigma_D^2+\sigma_S^2+\theta^2\sigma_t^2}\theta\right)\rho y_t + \kappa z_t. \quad \textbf{[S36]}$$

At this stage, it is convenient to introduce new variables that effectively map these equations to the same equations found for the particular model of the previous section. By defining

$$\tilde{\sigma}_S^2 \equiv \frac{\sigma_D^2+\sigma_S^2}{\theta^2}, \quad \tilde{\sigma}_H^2 \equiv \sigma_H^2 + \omega^2\left(\sigma_D^{-2}+\sigma_S^{-2}\right)^{-1}, \quad \tilde{\lambda} \equiv \lambda + \omega\theta\frac{\sigma_S^2}{\sigma_D^2+\sigma_S^2}, \quad \textbf{[S37]}$$

we thus have

$$\sigma_{t+1}^2 = \tilde{\lambda}^2\left(\sigma_t^{-2}+\tilde{\sigma}_S^{-2}\right)^{-1} + \tilde{\sigma}_H^2, \quad \textbf{[S38]}$$

which is formally identical to Eq. **S13**. Similarly, with

$$\tilde{x}_t \equiv \theta^{-1}(x_t - \rho y_t), \quad \textbf{[S39]}$$

we obtain

$$\ln W_t = \frac{1}{2}\ln\left(\frac{\tilde{\sigma}_S^2}{\tilde{\sigma}_S^2+\sigma_t^2}\right) - \frac{\left(m_t - \tilde{x}_t\right)^2}{2\left(\tilde{\sigma}_S^2+\sigma_t^2\right)} + \frac{1}{2}\ln\left(\frac{\sigma_S^2}{\sigma_S^2+\sigma_D^2}\right), \quad \textbf{[S40]}$$

which, up to an additive constant (which could be absorbed by redefining $r_{\max}$), is formally similar to Eq. **S14**. Finally, we can also write the recursion for m_t as follows:

$$m_{t+1} = \tilde{\lambda}\frac{\tilde{\sigma}_S^2}{\tilde{\sigma}_S^2+\sigma_t^2}m_t + \left(\tilde{\lambda}\frac{\sigma_t^2}{\tilde{\sigma}_S^2+\sigma_t^2} + \tilde{\kappa}\right)\tilde{x}_t + c_t, \quad \textbf{[S41]}$$

with

$$\tilde{\kappa} \equiv \left(\frac{\sigma_D^2+\rho\sigma_S^2}{\sigma_D^2+\sigma_S^2}\omega + \kappa\right)\frac{\theta}{1-\rho} \quad \textbf{[S42]}$$

and

$$c_t \equiv \left(\frac{\rho(\omega+\kappa)}{1-\rho}\right)b_t' + \kappa b_t''. \quad \textbf{[S43]}$$

Here again, we obtain, up to the addition of the random variable c_t, a relation for m_t that is formally similar to the one satisfied by the particular model, Eq. **S13**.

Given these correspondences, we can derive the long-term growth rate Λ of the general model from the expression found for the particular model, Eq. **S27**, without repeating the intermediate steps. This leads to the expression of Λ given by Eq. **S3**.

3. Optimization of Λ

3.1. Model of Fig. 1*A*. Starting from the expression for Λ given by Eq. **18** and assuming that its optimum is reached for a nonzero value of σ_H^2, we must have at this optimum:

$$\frac{\partial\Lambda}{\partial\alpha} = \frac{1}{2\alpha} - \frac{(1-a)(1+a\alpha^2)}{(1+\alpha)^2(1-a\alpha)^2}\sigma_E^2 = 0, \quad \textbf{[S44]}$$

that is,

$$\sigma_E^2 = \frac{(1+\alpha)^2(1-a\alpha)^2}{2(1-a)\alpha(1+a\alpha^2)}. \quad \textbf{[S45]}$$

This solution is valid until σ_H^2 reaches 0, i.e., α reaches 1. Substituting $\alpha=1$ in Eq. **S45** gives the following condition:

$$\sigma_E^2 > 2\frac{1-a}{1+a}. \quad \textbf{[S46]}$$

When this condition is not met, Λ is optimal for $\sigma_H^2=0$. These results are represented in Fig. 2.

3.2. Model of Fig. 3. The model depicted in Fig. 3 is defined from the general model by taking $\lambda=1$, $\theta=1$, $\kappa=0$, $\omega=0$, $\rho=0$, and $\sigma_S^2=1$. In this limit, Eq. **S3** becomes the following:

$$\Lambda = r_{\max} + \frac{1}{2}\ln\frac{\alpha}{1+\sigma_D^2} - \frac{(1-a)\alpha}{(1+\alpha)(1-a\alpha)}\frac{\sigma_E^2}{1+\sigma_D^2}, \quad \textbf{[S47]}$$

with

$$\alpha = \frac{2}{2+\tilde{\sigma}_H^2 + \left(\tilde{\sigma}_H^2\left(\tilde{\sigma}_H^2+4\right)\right)^{1/2}}, \qquad \tilde{\sigma}_H^2 = \frac{\sigma_H^2}{1+\sigma_D^2}. \quad \textbf{[S48]}$$

Instead of optimizing Λ over $(\sigma_H^2, \sigma_D^2) \in [0,\infty[\times[0,\infty[$, we can equivalently but more conveniently optimize

$$\tilde{\Lambda} = r_{\max} + \frac{1}{2}\ln\frac{\alpha}{1+\sigma_D^2} - \frac{(1-a)\alpha}{(1+\alpha)(1-a\alpha)}\frac{\sigma_E^2}{1+\sigma_D^2}, \quad \textbf{[S49]}$$

over $(\alpha, \sigma_D^2) \in [0,1]\times[0,\infty[$, where α is treated independently of σ_D^2: the two optimizations must give the same optimal value, $\max_{\sigma_H^2,\sigma_D^2}\Lambda(\sigma_H^2,\sigma_D^2) = \max_{\alpha,\sigma_D^2}\tilde{\Lambda}(\alpha,\sigma_D^2)$, and $\hat{\sigma}_D^2=0$ when optimizing Λ if and only if $\hat{\sigma}_D^2=0$ when optimizing $\tilde{\Lambda}$, as well as $\hat{\sigma}_H^2=0$ when optimizing Λ if and only if $\hat{\alpha}=1$ when $\tilde{\Lambda}$.

First, if we assume $\hat{\sigma}_D^2>0$, we must have $\partial\tilde{\Lambda}/\partial\sigma_D^2=0$, which gives the following:

$$1+\hat{\sigma}_D^2 = 2\frac{(1-a)\alpha}{(1+\alpha)(1-a\alpha)}\sigma_E^2. \quad \textbf{[S50]}$$

Assuming additionally $\hat{\alpha}<1$, we must have $\partial\tilde{\Lambda}/\partial\alpha=0$, which gives the following:

$$\hat{\alpha} = \frac{1-a}{2a}. \quad \textbf{[S51]}$$

For this solution with $(\hat{\sigma}_H^2>0, \hat{\sigma}_D^2>0)$ to hold, we need $(1-a)/(2a)<1$, i.e., $a>1/3$. We thus conclude that the boundary between $(\hat{\sigma}_H^2>0, \hat{\sigma}_D^2>0)$ and $(\hat{\sigma}_H^2=0, \hat{\sigma}_D^2>0)$ is given by $a=1/3$ (black line in Fig. 4*E*).

By the same argument of continuity, the boundary between $(\hat{\sigma}_H^2>0, \hat{\sigma}_D^2>0)$ and $(\hat{\sigma}_H^2>0, \hat{\sigma}_D^2=0)$ corresponds to Eq. **S50** having for solution $\hat{\sigma}_D^2=0$, which gives $\sigma_E^2=((1+a)/(1-a))^2/4$ (blue curve in Fig. 4*E*).

If we now assume $\hat{\sigma}_D^2=0$ but $\hat{\sigma}_H^2>0$, we obtain an equation for $\hat{\alpha}$ from $\partial\tilde{\Lambda}(\sigma_D^2=0)/\partial\alpha=0$. Substituting $\alpha=1$ in this equation gives the boundary between $(\hat{\sigma}_H^2>0, \hat{\sigma}_D^2=0)$ and $(\hat{\sigma}_H^2=0, \hat{\sigma}_D^2=0)$, namely $\sigma_E^2=2(1-a)/(1+a)$ (red curve in Fig. 4*E*).

Finally, assuming $\hat{\sigma}_H^2=0$ but $\hat{\sigma}_D^2>0$, we obtain an equation for $\hat{\sigma}_D^2$ from $\partial\tilde{\Lambda}(\alpha=1)/\partial\sigma_D^2=0$, which gives $\hat{\sigma}_D^2=\sigma_E^2-1$. The boundary between $(\hat{\sigma}_H^2=0, \hat{\sigma}_D^2>0)$ and $(\hat{\sigma}_H^2=0, \hat{\sigma}_D^2=0)$ is therefore given by $\sigma_E^2=1$ (green line in Fig. 4*E*).

All together, we thus obtain the diagram presented in Fig. 4*E*, with boundaries described by Eq. **22** in the main text. The corresponding analytical expressions for the optimal values of σ_H^2 and σ_D^2 (shown in Fig. 4 *B* and *C*) are as follows:

$$\left(\hat{\sigma}_H^2>0, \hat{\sigma}_D^2>0\right):\ \hat{\sigma}_H^2 = \frac{2(1-a)(1-3a)^2}{a(1+a)^2}\sigma_E^2, \qquad \hat{\sigma}_D^2 = 4\left(\frac{1-a}{1+a}\right)^2\sigma_E^2 - 1, \quad \textbf{[S52]}$$

$$\left(\hat{\sigma}_H^2=0, \hat{\sigma}_D^2>0\right):\ \hat{\sigma}_H^2=0, \quad \hat{\sigma}_D^2=\sigma_E^2-1, \quad \textbf{[S53]}$$

$$\left(\hat{\sigma}_H^2>0, \hat{\sigma}_D^2=0\right):\ \hat{\sigma}_H^2=\hat{\varsigma}_H^2(a,\sigma_E^2), \quad \hat{\sigma}_D^2=0, \quad \textbf{[S54]}$$

where $\hat{\varsigma}_H^2(a,\sigma_E^2)=(1-\alpha)^2/\alpha$, with α solution of Eq. **S45**.

3.3. Model of Fig. 3 with Directed Environmental Changes. When the environmental process is of the form $x_t=ct+b_t$ with $b_t\sim\mathcal{N}(0,\sigma_E^2)$, the expression for the growth rate of the model of Fig. 3 becomes the following:

$$\Lambda = \frac{1}{2}\ln\left(\frac{\alpha}{1+\sigma_D^2}\right) - \frac{\alpha}{2(1-\alpha)^2}\frac{c^2}{1+\sigma_D^2} - \frac{\alpha}{1+\alpha}\frac{\sigma_E^2}{1+\sigma_D^2}, \quad \textbf{[S55]}$$

where α is given by Eq. **S48** (we assume here that $r_{\max}=0$ and $\sigma_S^2=1$). Fig. 4*F* presents the results of an optimization of this expression over (σ_H^2,σ_D^2) and as a function of (c,σ_E^2).

3.4. Model of Fig. 5. As an alternative to the optimization over $(\lambda,\omega,\sigma_H^2)$ for all $\lambda\geq 0$, $\omega\geq 0$, and $\sigma_H^2\geq 0$, we can restrain the optimization to $\lambda\in\{0,1\}$, $\omega\in\{0,1\}$, and $\sigma_H^2\geq 0$. The results, presented in Fig. S2, are qualitatively similar.

3.5. Models of Fig. 7. Separate numerical optimizations of the models P and L for $\sigma_I^2=\sigma_H^2=\sigma_D^2=1$ lead to the results presented in Fig. S3. The comparison between the growth rate Λ of the two models is presented in Fig. S3*C*, which thus corresponds to the first panel of Fig. 8.

4. Numerical Simulations of Population Dynamics

4.1. Algorithm. To perform the numerical simulations of the results presented in Fig. 2*C*, we start from a population of N individuals i, each characterized by two parameters, γ_i and σ_{Hi}. We then iteratively compute the environment and the composition of the population: at each time step t, we compute the state of the environment x_t as $x_t\sim\mathcal{N}(ax_{t-1},\sigma_E^2)$, and for each individual i in the population, we compute their attributes γ_i', σ_{Hi}', and their number of offsprings ξ_i as follows:

$$\gamma_i' \sim \mathcal{N}\left(\gamma_i, \sigma_{Hi}^2\right), \quad \sigma_{Hi}' \sim \max\left(0, \mathcal{N}\left(\sigma_{Hi}, \sigma_M^2\right)\right), \qquad \xi_i \sim \mathcal{P}\left(\exp\left(r_{\max} - (\gamma_i - x_t)^2/\left(2\sigma_S^2\right)\right)\right), \quad \textbf{[S56]}$$

where $\mathcal{N}(m,\sigma^2)$ denotes a normal distribution of mean m and variance σ^2, and $\mathcal{P}(m)$ a Poisson distribution of mean m. We thus obtain $N'=\sum_i\xi_i$ offsprings with an associated multiplicative factor $W_t=N'/N$. If $N'\leq N$, we take all of the offsprings to define the next generation; otherwise, we draw randomly N of them to limit the total population size to at most N.

The growth rate is computed as $\Lambda=\sum_{t=\tau+1}^{\tau+T}\ln W_t/T$, where τ represents an initial relaxation time and $\tau+T$ the total number of generations that are simulated. The population may become extinct before the $\tau+T$ generation, but the parameter $r_{\max}$ can be augmented to limit this possibility. We also note that the choice of a Poisson distribution for ξ_i is just one of several possible choices: any distribution with the same mean should give equivalent results for Λ, although different choices may have different finite-size effects and, in particular, may be associated with different probabilities of extinction.

Fig. 2*C* was obtained using $a=0.5$, $\sigma_X^2=(1-a^2)\sigma_E^2=0.75$, $\sigma_M^2=10^{-5}$, $r_{\max}=4$, $N=10^3$, $\tau=2,500$, and $T=10^4$. The initial conditions were $\gamma_i\sim\mathcal{N}(0,1)$ and $\sigma_{Hi}=0.5$ for all i (a value much higher than the optimal value $\hat{\sigma}_H^2$). The data displayed in Fig. 2*C* are combined population and time averages of σ_{Hi}^2 over the T generations of a single simulation, with error bars representing SDs. The results show that σ_{Hi}^2 fluctuates around the value $\hat{\sigma}_H^2$ that optimizes the growth rate Λ (red curve in Fig. 2*C*).

Fig. S1 *A–C* presents the results obtained when repeating the same simulations with different values of σ_M^2: $\sigma_M^2=10^{-3}$ (*A*), $\sigma_M^2=10^{-4}$ (*B*), and $\sigma_M^2=10^{-5}$ (*C*) (with smaller values of σ_M^2, $T=10^4$ generations are not sufficient to reach a stationary regime when starting from $\sigma_{Hi}=0.5$ for all i).

4.2. Non-Gaussian Distributions. To verify that our results are qualitatively reproduced with non-Gaussian distributions, we repeated the simulations performed for Fig. 2*C* taking the selection, the mutations, or the environmental fluctuations to follow a Laplace distribution $\mathcal{L}(m,\sigma^2)$ instead of normal distribution $\mathcal{N}(m,\sigma^2)$. The Laplace distribution $\mathcal{L}(m,\sigma^2)$ with mean m and variance σ^2 is defined from the probability density $L(x)=e^{-2|x-m|/\sigma}/\sigma$. In Fig. S1*D*, we thus take $\xi_i\sim\mathcal{P}(\exp(r_{\max}-|\gamma_i-x_t|/\sigma_S))$ instead of $\xi_i\sim\mathcal{P}(\exp(r_{\max}-(\gamma_i-x_t)^2/(2\sigma_S^2)))$, in Fig. S1*E*, $\gamma_i'\sim\mathcal{L}(\gamma_i,\sigma_{Hi}^2)$ instead of $\gamma_i'\sim\mathcal{N}(\gamma_i,\sigma_{Hi}^2)$, and in Fig. S1*F*, $x_t\sim\mathcal{L}(ax_{t-1},\sigma_E^2)$ instead of $x_t\sim\mathcal{N}(ax_{t-1},\sigma_E^2)$; all of the other parameters are otherwise as in Fig. 2*C*.

In the last case where only the distribution of the environmental fluctuations is affected, the population dynamics is still described exactly by our model: the derivation of Λ indeed involves only the variance σ_E^2 of b_t (where $x_t=ax_{t-1}+b_t$) and is not dependent on its tail. However, we observe when taking the same parameters as in Fig. 2*C* that the population has a much higher probability of being extinct than with Gaussian fluctuations (explaining the absence of data at low a in Fig. S1*E*). This illustrates a limitation of a description based solely on the long-term growth rate Λ: it does not capture the fluctuations that may lead a finite population to extinction.

5. Derivation of Price Equation

We consider here the simple model of Fig. 1*A*, described by Eqs. **1** and **6**, i.e.,

$$W_t\, n_{t+1}(\gamma')=\sum_\xi \xi\int \mathrm{d}\gamma\, R(\xi|\gamma,x_t)\, H(\gamma'|\gamma) n_t(\gamma), \qquad \textbf{[S57]}$$

where we do not make here any assumption on $R(\xi|\gamma,x_t)$ and $H(\gamma'|\gamma)$, besides the fact that they are stochastic kernels. We adopt the usual notation for writing Price equation, and denote the mean trait in the population, m_t, by $\langle\gamma\rangle_t$ and the mean "instantaneous fitness," W_t, by $\langle\xi\rangle_t$. The mean change of the trait between two generations, which is the focus of Price equation, is denoted by $\Delta_t\langle\gamma\rangle\equiv\langle\gamma\rangle_{t+1}-\langle\gamma\rangle_t$, and the covariance between the trait and the instantaneous fitness, which is the other central quantity, is defined by $\mathrm{cov}_t(\xi,\gamma)=\langle\xi\gamma\rangle_t-\langle\xi\rangle_t\langle\gamma\rangle_t$, where the subscript t accounts for the time dependence of these quantities in our model.

Using the identity $\int\mathrm{d}\gamma\, H(\gamma|\gamma')=1$, we first rewrite Eq. **S57** as follows:

$$\begin{aligned} W_t\, n_{t+1}(\gamma') = &\sum_\xi \xi\, R(\xi|\gamma',x_t) n_t(\gamma') \\ &+\sum_\xi \xi\int \mathrm{d}\gamma \big[R(\xi|\gamma,x_t)H(\gamma'|\gamma)n_t(\gamma) \\ &\qquad - R(\xi|\gamma',x_t)H(\gamma|\gamma')n_t(\gamma')\big]. \end{aligned} \qquad \textbf{[S58]}$$

After multiplying by γ' and integrating over this variable, we thus obtain the following:

$$W_t\langle\gamma\rangle_{t+1}\equiv W_t\int\mathrm{d}\gamma'\,\gamma'\, n_{t+1}(\gamma')=\langle\xi\gamma\rangle_t+\langle\xi\Delta\gamma\rangle_t, \qquad \textbf{[S59]}$$

where $\Delta\gamma$, a function of (γ,ξ,x_t), is defined by the following:

$$\Delta\gamma\equiv\gamma\int\mathrm{d}\gamma'\,\left[R(\xi|\gamma',x_t)\, H(\gamma|\gamma')\, n_t(\gamma')-R(\xi|\gamma,x_t)\, H(\gamma'|\gamma)\, n_t(\gamma)\right]. \qquad \textbf{[S60]}$$

Finally, subtracting $W_t\langle\gamma\rangle_t=\langle\xi\rangle_t\langle\gamma\rangle_t$ to both sides of Eq. **S59** leads to the following:

$$\langle\xi\rangle_t\,\Delta_t\langle\gamma\rangle=\mathrm{cov}_t(\xi,\gamma)+\langle\xi\Delta\gamma\rangle_t, \qquad \textbf{[S61]}$$

which is the usual form of Price equation. For our simple model, this formula decomposes evolutionary change in two terms: $\mathrm{cov}_t(\xi,\gamma)$, the contribution from selection, and $\langle\xi\Delta\gamma\rangle_t$, the contribution from mutations, which vanishes when transmission is error free, i.e., when $H(\gamma'|\gamma)=\delta(\gamma'-\gamma)$.

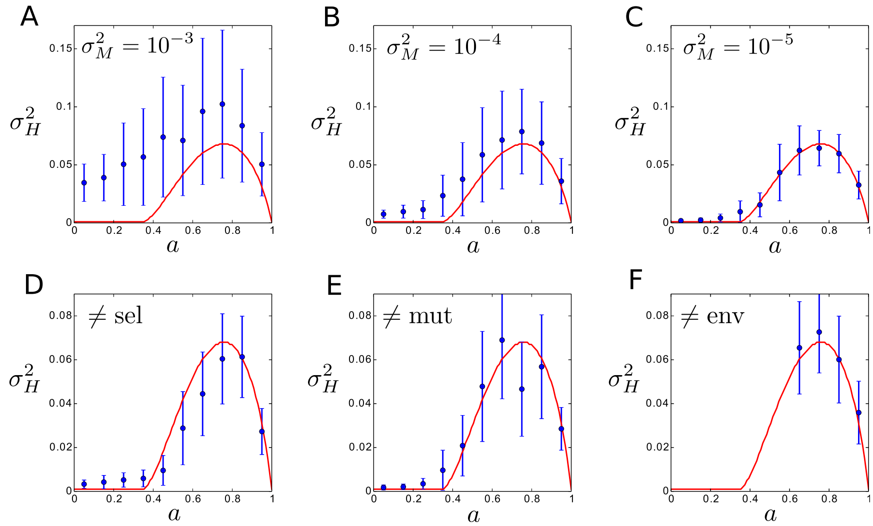

Fig. S1. *C* is identical to Fig. 2*C*. *A* and *B* correspond, respectively, to taking $\sigma_M^2 = 10^{-3}$ and $\sigma_M^2 = 10^{-4}$ instead of $\sigma_M^2 = 10^{-5}$. *D–F* correspond, respectively, to taking a Laplace distribution for the selection, mutations, and environmental fluctuations instead of Gaussian distributions; in the last case, the missing data for $a < 0.6$ correspond to instances where the population became extinct before the end of the simulation (*SI Appendix*).

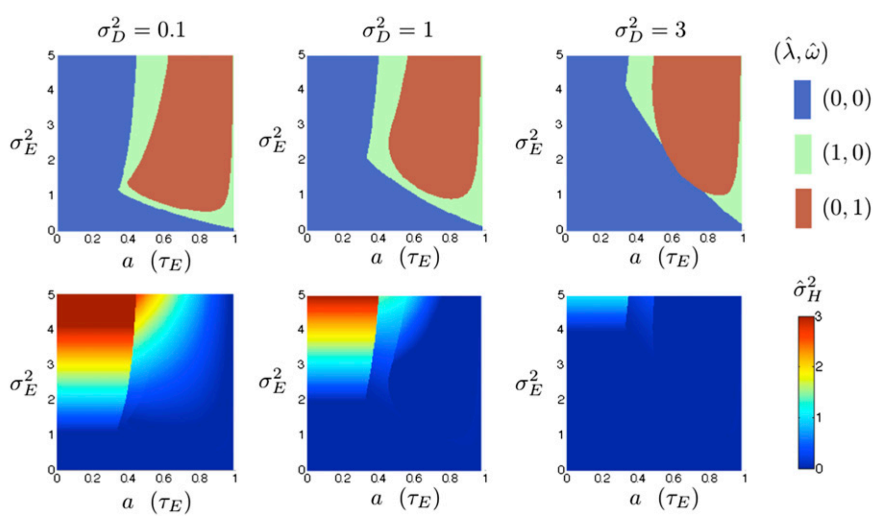

Fig. S2. When to separate phenotype and transmitted genotype? For the model of Fig. 5, results of optimizations of Λ over $(\lambda, \omega, \sigma_H^2)$ with $(\lambda, \omega) \in \{0,1\}^2$, for three fixed values of σ_D^2.

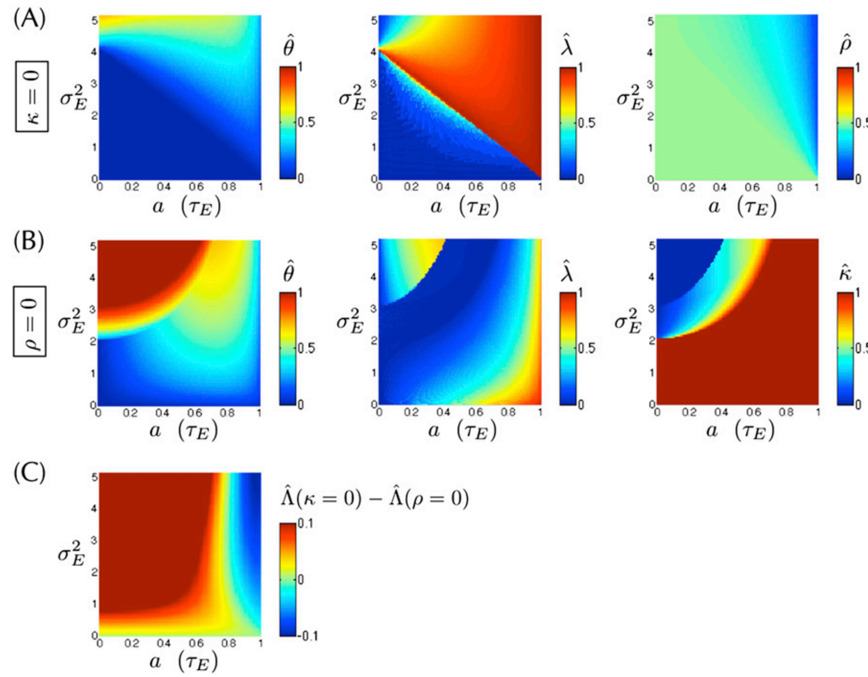

Fig. S3. Where to acquire information? (*A*) Model P: Phase diagram for the optimal values of (θ,λ,ρ) as a function of the environmental variables (a,σ_E^2) for $\sigma_H^2=1$, $\sigma_D^2=1$, $\sigma_{Ip}^2=1$, and $\kappa=0$. (*B*) Model L: Phase diagram for the optimal values of (θ,λ,κ) as a function of the environmental variables (a,σ_E^2) for $\sigma_H^2=1$, $\sigma_D^2=1$, $\sigma_{Il}^2=1$, and $\rho=0$. (*C*) Difference between the optimal growth rates of model P, $\hat{\Lambda}(\kappa=0)$, and model L, $\hat{\Lambda}(\rho=0)$.